AN ANALYTIC APPROACH TO OPTIMIZATION OF TIDAL TURBINE ARRAYS

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SUMMARY: Motivated by global warming due to CO_2-emission various technologies for harvesting of energy from renewable sources are developed. Hydrokinetic turbines get applied to surface watercourse or tidal flow to gain electrical energy. Since the available power for hydrokinetic turbines is proportional to the projected cross section area, fields of turbines are installed to scale shaft power. Each hydrokinetic turbine of a field can be considered as a disk actuator. In [1], the first author derives the optimal operation point for hydropower in an open-channel. The present paper concerns about a 0-dimensional model of a disk-actuator in an open-channel flow with bypass, as a special case of [1]. Based on the energy equation, the continuity equation and the momentum balance an analytical approach is made to calculate the coefficient of performance for hydrokinetic turbines with bypass flow as function of the turbine head and the ratio of turbine width to channel width.

Keywords: low head hydropower, open surface, tidal turbine arrays

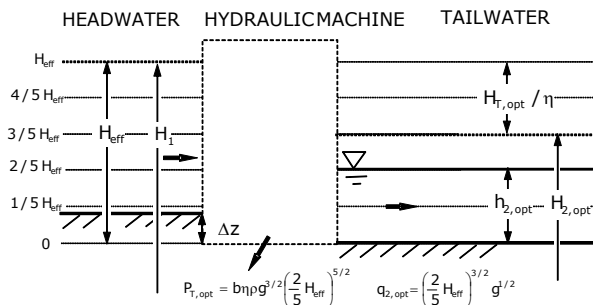## 1. Optimal Operation Point for Open-Channel flows

Figure 1: Optimal operation for hydropower in an open-channel flow [1].

The paper „Upper Limit for Hydropower in an Open-Channel Flow" by the first author [1], deals with hydropower as an optimization problem. Based on strictly axiomatic derivation, the optimal flow rate and optimal head are determined, using the energy balance. Figure 1 shows the result of [1], i.e. the optimal operating point. The optimal head is $2\eta/5$ (being η the hydraulic efficiency) of the effective head $H_{eff} := \Delta z + E_1$ for given upstream energy height $E_1 := h_{10} + u_{10}^2/2g$. The coefficient of performance is defined as usual [1]

$$C_p := \frac{P_T}{P_{avail}} \quad . \qquad (1)$$

P_T is the hydraulic power gained by the turbine

$$P_T = \eta \, Q \, \rho g H_T. \qquad (2)$$

$q_2 = Q/b$ is the flow rate per depth unit for the tailwater width b of an open channel.

The available power P_{avail} represents the power that can be gained by a hypothetical ideal machine at its best operation point. It is defined as [1]
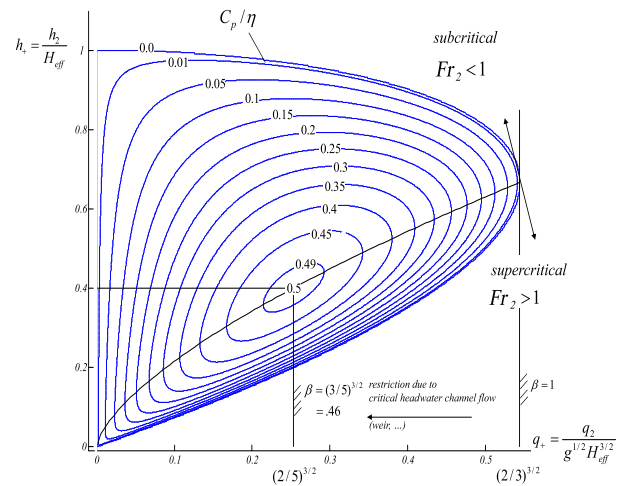

Figure 2: Coefficient of performance as a function of dimensionless volume flow rate and head [1].

$$P_{avail} := 2\rho b \left(\frac{2}{5}H_{eff}\right)^{5/2} g^{3/2}. \qquad (3)$$

With (1), (2) and (3) the coefficient of performance C_p can be written as a function of the operating point, with a maximum of $C_{p,max} = \eta/2$. I.e. even for a hydraulic efficiency of 100%, max. 50% of the available hydropower can be transformed into mechanical power in the best operation point sketched in Figure 2.

Figure 2 shows the coefficient of performance as function of the dimensionless water depth of the tailwater $h_+ := h_2/H_{eff}$ and the dimensionless specific volume flow rate $q_+ := Q/(bg^{1/2}H_{eff}^{3/2})$. The maximum of C_p/η determines the optimal operating point, with the optimal head

$$H_{T,opt} = \frac{2}{5}\eta H_{eff} \quad , \qquad (4)$$

the optimal volume flow rate [1]

$$Q_{opt} = b \left(\frac{2}{5}\right)^{3/2} g^{1/2} H_{eff}^{3/2} \quad , \qquad (5)$$

and the optimal water depth level in the tailwater

$$h_{2,opt} = \frac{2}{5} H_{eff} \ . \qquad (6)$$

Thus the optimal turbine power is

$$P_{T,opt} = \rho g H_{T,opt} Q_{opt} = \eta b \rho g^{3/2} \left(\frac{2}{5} H_{eff} \right)^{5/2} . \ (7)$$

The result $C_P/\eta \leq 0.5$ is independent of a specific machine design. Any special case with its associated theory has to be consistent to the above results. This consistency is shown to be valid for the special case of hydrokinetic turbines in the following section.

2. Application to Open-Channel with Bypass

Nowadays hydrokinetic turbines are applied to surface or tidal watercourses without damming the water. In most cases the width of the turbine is less than the width of the channel to which it gets applied. Thus there will be a part of the volume flow passing beside the turbine, which cannot be used for the gaining of energy and therefore has to be considered as leakage.

Then two limiting cases (i) a turbine of infinite width [2] and (ii) a turbine of infinite height. The later case was not considered in depth up to now and is the subject of this paper. More specific, issue of the present paper, is a 0-dimensional Model for a disc actuator of infinite height, as shown in Figure 3.A.

The question to be answered is:

What is the influence of the dimensionless turbine head $H_T/\eta H_1$ and the dimensionless turbine width σ on the coefficient of performance?

The 0-dimensional model considers the flow velocity and water level height at 6 points (see Fig.3, 1, +, −, o, i, 2) and the stream tube contraction ratio in the points 1 and i. Due to continuity of the stress tensor, the water level heights of the points i,o have to be of equal magnitude. This leads to the 13 unknowns $u_1, h_1, u_+, h_+, u_-, h_-, u_i, u_o, h_{i,o}, u_2, h_2, \beta_+, \beta_-$.

By subdividing the model into the control-volumes shown in Figure 3.B, one can derive five equations based on the conservation of energy and 5 equations for the conservation of mass, one for each control-volume. Furthermore three equations can be derived based on the momentum balance. This leads to the nonlinear system of equations shown Table. 1.

Solving the system of equations with analytic methods is not possible. Commercial numerical optimization tools, as provided by Matematica or Matlab, did not deliver satisfactory results. The reason for this is the nonlinearity of the algebraic system of 13 unknowns with a polynomial degree of 3. The maximum number of solutions can be up to $3^{13} \approx 10^6$. Hence, to solve the system of equations, an optimization algorithm was developed and programmed, based on the principle of least square methods. A more specific description of the optimization algorithm is given in the following section.

3. Description of Optimization Algorithm

The optimization algorithm calculates the least square error of the system of equations (Table 1) for a given pair

of parameter values β_- and h. Within a nested loop iteration it varies the water level height h within the range $\{0, ..., 1 - H_T/H_1\eta\}$ with number of steps n and the contraction ratio β_- within the range $\{0, ..., 1\}$ with the number of steps m.

For a better understanding of the optimization algorithm, Figure 4 shows the process diagram for the flow considered in the above section. The dashed lines represent the conservation of mass law in a plane of dimensionless water level height $\tilde{h}$ and flow velocity $\tilde{u}$. The solid lines represent isolines of energy, hence the conservation of energy. Since all physically relevant solutions of the system of equations must satisfy both, conservation of energy and conservation of mass law, all points, representing the state of flow, must be intersection points of dashed lines with solid lines. In this case physically relevant means: $0 < \tilde{h} < 1$, $0 < \tilde{u} < 1$, $Im(\tilde{h}) = Im(\tilde{u}) = 0$ For a given pair of parameter values $\beta_{-,m}$ and h_n the intersection points are calculated. Since the conservation of mass and the conservation of energy intersect twice, there is one intersection point with a subcritical and one with a supercritical flow condition. With the intersection point method $m \times n$ sets of parameter values for the 13 unknowns can be generated, satisfying equations (1), (2), (3), (4), (6), (7), (8), (9) and (13) of Table 1. Equations (10), (11), and (12) will only be satisfied for one pair of h and β_-. For all other pairs of h and β_- the right side of those equations will not vanish. Instead there will be an scalar value on the right side, which can be considered as error of the optimization algorithm.

During the iteration the solver records the square of the sum of the errors in a $m \times n$ – Matrix. The set of parameter values that leads to the minimal value for this Matrix is identified as solution of the system of equations.
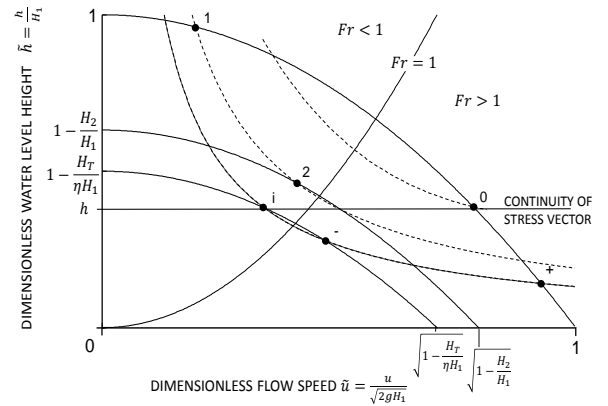

Figure 3: Process diagram for open-channel flow with bypass

4. Results

The answer to the question raised in the section 2, can be read from Figure 5. For $\sigma = 0$, or in other terms for an empty channel without any disk-actuator, there is no mechanical power gained. Hence the coefficient of performance vanishes. For $\sigma = 1$, or in other terms, a disk-actuator of the same width as the channel without bypass, the results are consistent to the results of [1] as expected. For $H_T/\eta H_1 = 0.4$ the optimum for the

coefficient of performance $C_p/\eta = 0.5$ is reached. For Figure 5.A with values of $0 < H_T/\eta H_1 < 0.4$ the maximum coefficient of performance is increasing from $C_p/\eta = 0$ to $C_p/\eta = 0.5$. For $0 < H_T/\eta H_1 < 0.3$ the curves do not increase monotonically but do have their maximum for values of $\sigma \neq 1$. For Figure 5.B with values of $0.4 \leq H_T/\eta H_1 < 1$ the maximum coefficient of performance is decreasing from $C_p/\eta = 0.5$ to $C_p/\eta = 0$. For values of $H_T/\eta H_1$ and σ near 1 or zero the uncertainty of the solver rises. Hence for these values of σ and $H_T/\eta H_1$ the resolution of the optimization problem has to be improved. Yet the development of the optimization algorithm is in progress.
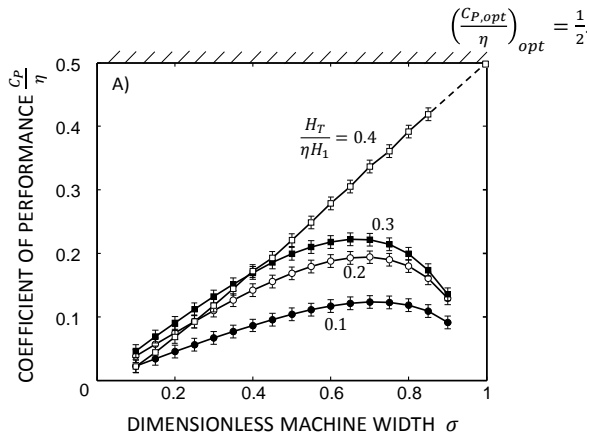

Figure 5.A: Coefficient of Performance with $0 < H_T/\eta H_1 < 0.4$
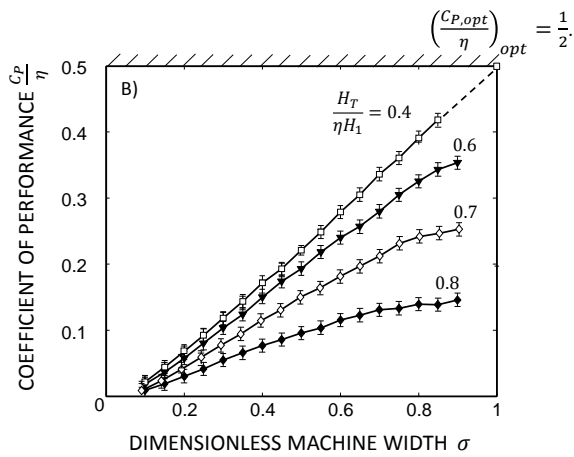

Figure 5.B: Coefficient of Performance with $0.4 < H_T/\eta H_1 < 0.8$

5. Conclusions

The nonlinear system of equations describing a disk-actuator is described and discussed in detail for the first time. It can be solved by the developed optimization algorithm. Knowing the wake conditions of one disk-actuator, fields of turbines can be described with parallel or series connections of the disk-actuator model.

Since the optimization problem considered by [1] is valid independently of the machine type installed to an open-channel flow, the results of the disk-actuator with

bypass have to be consistent to the results of Pelz. The 0-dimensional model accomplishes this consistence.

Using the results of this paper, the coefficient of performance can be predicted for fields of hydrokinetic turbines in tidal current or surface watercourse applications.

[1]   Pelz, P: Upper Limit for Hydropower in an Open-Channel Flow, Journal of Hydraulic Engineering; ASCE / November (2011); DOI: 10.1061/(ASCE)HY.1943-7900.0000393

[2]   Wheelan, J. et al.: A free-surface and blockage correction for tidal turbines, Journal of Fluid Mechanics vol. 624; Cambridge University Press (2009); DOI: 10.1017/S002211200900591
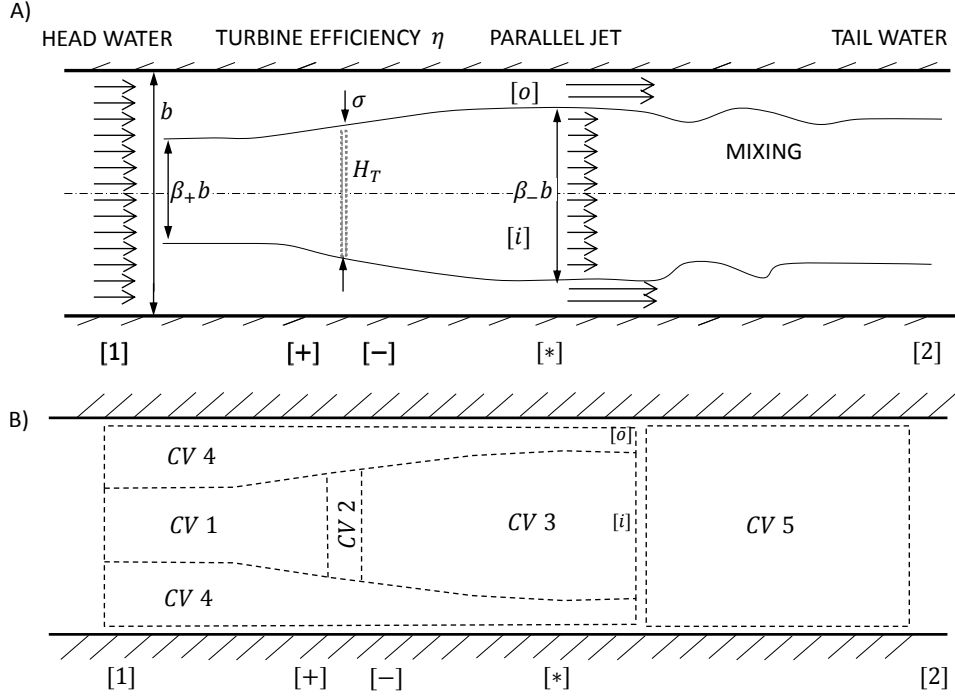
Figure 4: 0-dimensional disc-actuator model for tidal turbines.

Control Volume	Equation	Type
$CV\ 1$:	$u_1 h_1 \beta_+ - u_+ h_+ \sigma = 0$	
$CV\ 2$:	$u_+ h_+ - u_- h_- = 0$	
$CV\ 3$:	$u_- h_- \sigma - u_i h \beta_- = 0$	*CONSERVATION OF MASS*
$CV\ 4$:	$u_1 h_1 (1 - \beta_+) - u_o h (1 - \beta_-) = 0$	
$CV\ 5$:	$u_o h (1 - \beta_-) + u_i h \beta_- - u_2 h_2 = 0$	
$CV\ 1$:	$H_1 - \left(h_+ + \frac{u_+^2}{2g}\right) = 0$	
$CV\ 2$:	$h_+ + \frac{u_+^2}{2g} - \left(h_- + \frac{u_-^2}{2g}\right) - \frac{H_T}{\eta} = 0$	
$CV\ 3$:	$h_- + \frac{u_-^2}{2g} - \left(h + \frac{u_i^2}{2g}\right) = 0$	*CONSERVATION OF ENERGY*
$CV\ 4$:	$H_1 - \left(h + \frac{u_o^2}{2g}\right) = 0$	
$CV\ 5$:	$u_i h \beta_- \left(h + \frac{u_i^2}{2g}\right) + u_o h (1 - \beta_-) \left(h + \frac{u_o^2}{2g}\right) - u_2 h_2 \left(h_2 + \frac{u_2^2}{2g}\right) - h_{l,mix} = 0$	
$[1] - [2]$:	$h_1 \left(\frac{h_1}{2} + \frac{u_1^2}{g}\right) - h_2 \left(\frac{h_2}{2} + \frac{u_2^2}{g}\right) - \frac{h_1 H_T \beta_+}{\eta} = 0$	*CONSERVATION OF MOMENTUM*
$CV\ 5$:	$h(1 - \beta_-) \left(\frac{h}{2} + \frac{u_o^2}{g}\right) + h \beta_- \left(\frac{h}{2} + \frac{u_i^2}{g}\right) - h_2 \left(\frac{h_2}{2} + \frac{u_2^2}{g}\right) = 0$	
$[1]$:	$H_1 - \left(h_1 + \frac{u_1^2}{2g}\right) = 0$	*DEFINITION OF HEAD*

Table 1: Equation System for 2D Open Channel-Flow with bypass